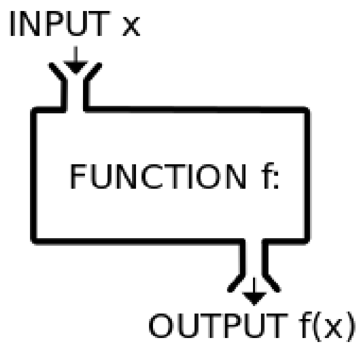
Domain and Range

The **domain** of a function is the set of values that we are allowed to plug into our function. This set is the x values in a function such as $f(x)$.

The **range** of a function is the set of values that the function assumes. This set is the values that the function shoots out after we plug an x value in. They are the y values.

We can think of a function as a machine along an assembly line. On one end of the assembly line we have a few screws and bolts and on the other end we have a car. The machine in the middle is the function. The screws and bolts that we input into the machine (our function) is the domain. The car (or output) at the other can be thought of as the range.



Many problems will ask you to find the domain of a function. What does this mean?

All the problem is asking you is to find what values of x can be plugged into the function. This is useful to know since some functions have limits on what is permissible as an input. For instance, consider the function:

$$f(x) = \frac{1}{x} \tag{1}$$

We know that we can never divide by 0 so here our domain can not include the value $x = 0$. However, all other values of x would be OK. We can plug in any other number into our function and we would get an output. If we ever put a number into a function and we can't get an output then we know that there is some sort of domain issue. The most common occurrences of this happen with:

- dividing by 0
- negative square roots
- negative logs

There are two main ways to write domains: **interval notation** and **set notation**.

Interval notation used parenthesis or brackets to imply where the function is defined. In the case of our example, we would write our domain using interval notation in the following way:

$$D : (-\infty, 0) \cup (0, \infty) \tag{2}$$

All this is saying is from negative infinity up to 0 we can plug anything into our function and (the $\cup$ is called a union and it means 'and') from 0 (*but not including 0*) to positive infinity we can plug in anything. But, again, *not 0*.

Set notation uses sets to say explicitly where the function is or isn't defined. For instance, for our example we would use set notation in the following way:

$$D : \{x | x \neq 0\} \tag{3}$$

This can be read as D is our domain of all values of x such that (the vertical line means 'such that') x is not 0. Everything else is OK.

Before we look at some examples, lets talk for a little bit about **range**. Range is a little trickier to find than domain. Most of the time, we're going to have to look at the graph of the function to determine its range.

Examples

- **Example 1**

$$g(x) = \frac{6x - 2}{3x - 4} \tag{4}$$

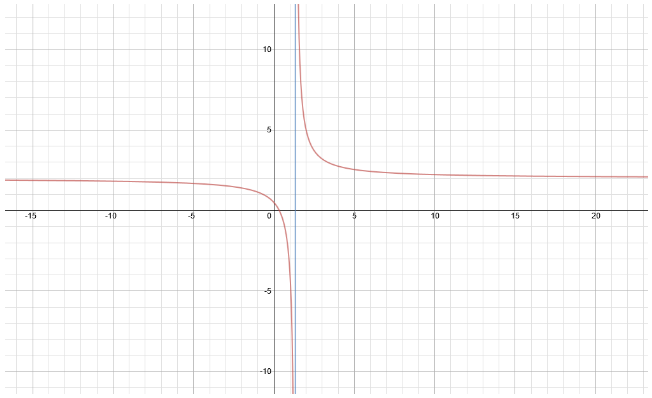
We obviously don't have any logs or square roots in this function so those two things won't cause any issues. We do have a fraction though and we know that we can never divide by 0. Therefore we will set the denominator of $g(x)$ equal to 0 and solve for x. This value(s) of x will be where our domain does not exist.

$$3x - 4 = 0 \tag{5}$$
$$3x = 4 \tag{6}$$
$$x = \frac{4}{3} \tag{7}$$

We just solved where our function is not defined. If we plug $x = \frac{4}{3}$ into our function we get a 0 in the denominator. We can write our domain in either of the following two ways:

$$D : \left(-\infty, \frac{4}{3}\right) \cup \left(\frac{4}{3}, \infty\right) \tag{8}$$
$$D : \{x | x \neq \frac{4}{3}\} \tag{9}$$

To find the range of this function we will have to look at the graph.



It should be fairly apparent that there is a horizontal asymptote at $y = 2$. This can also be calculated by dividing the coefficients of the leading terms of the numerator and denominator. This leads to $\frac{6}{3} = 2$. Thus, we can write our range in the following two ways:

$$R : (-\infty, 2) \cup (2, \infty) \tag{10}$$
$$R : \{y | y \neq 2\} \tag{11}$$

Notice: The function is graphed in red and the vertical line in blue is our asymptote where our domain gap is as well.

- **Example 2**

$$h(x) = \sqrt{x - 4} \tag{12}$$

We don't have any fractions or logs here but we do have a square root and we know that square roots can never be less than 0. (they can however be equal to 0) So we will set what's inside the square root to be greater than or equal to 0 and solve for x. Those values of x will be where our function is defined.

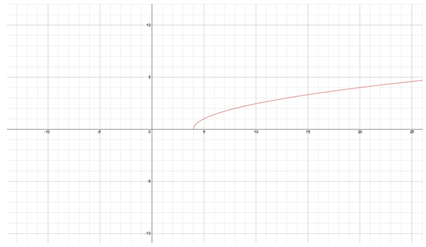
$$x - 4 \geq 0 \tag{13}$$
$$x \geq 4 \tag{14}$$

Subsequently our domain is:

$$D : [4, \infty) \tag{15}$$
$$D : \{x | x \geq 4\} \tag{16}$$

Notice: We use a closed bracket [instead of (to imply that the number 4 is permissible. We can plug 4 into our function and take the square root of 0 which is completely OK. (it equals 0)

To find the range we think about what all square root graphs look like. The graph for this function is:



We can then see that the range of this function will be:

$$R : [0, \infty) \tag{17}$$
$$R : \{y | y \geq 0\} \tag{18}$$

- **Example 3**

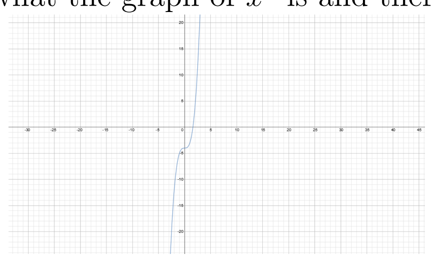
$$r(x) = x^3 - 4 \tag{19}$$

This problem is a little different in that it doesn't have any fractions, square roots or logs. It also doesn't appear to have any values of x that will make the function undefined. Thus, we say it has an infinite domain. Thus we write the domain as:

$$D : (-\infty, \infty) \tag{20}$$
$$D : \{x | x \in \mathbb{R}\} \tag{21}$$

Note: the $\in$ means 'an element of' and therefore we are saying x can be any real number $\mathbb{R}$.

For the range, we know what the graph of x^3 is and therefore our function looks like:



We then see that the range is also infinite and we write the range as:

$$R : (-\infty, \infty) \tag{22}$$
$$R : \{y | y \in \mathbb{R}\} \tag{23}$$

Here are some example problems for you to work out on your own with their respective answers at the bottom:

Find the domains and ranges of the following functions.

$$m(x) = \frac{2x}{2x-2} \tag{24}$$
$$a(x) = \sqrt{x^2-4} \tag{25}$$
$$t(x) = \sqrt{x^2+4} \tag{26}$$
$$h(x) = \frac{x^2-1}{x-1} \tag{27}$$
$$s(x) = \frac{\sqrt{x-4}}{x^2-25} \tag{28}$$

answers for domain in order: $(-\infty, 1) \cup (1, \infty)$ $(-\infty, -2) \cup (2, \infty)$ $(-\infty, \infty)$ $(-\infty, 1) \cup (1, \infty)$ $[4, 5) \cup (5, \infty)$

answers for range in order: $(-\infty, 1) \cup (1, \infty)$ $(0, \infty)$ $(0, \infty)$ $(-\infty, 1) \cup (1, \infty)$ $(-\infty, \infty)$