Exponents

Say we wanted to multiply 2 by itself 10 times. We could write it as,

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 1024$$

However, this is clearly too long and it's not immediately clear as to how many 2's we actually are multiplying together. And what if we have to multiply 100 2's together? It would just be entirely too long. Luckily we have another notation that allows us to show this much more efficiently.

If we wanted to multiply two by itself 10 times we could use exponential notation to write it as,

$$2^{10} = 1024$$

Often times we wish to simply exponents. Here are the most common rules for simplifying exponents, some of which we will derive to show why they are true. In these rules, m and n are just numbers.

$$x^m x^n = x^{m+n} \tag{1}$$

$$\frac{x^m}{x^n} = x^{m-n} \tag{2}$$

$$(xy)^n = x^n y^n \tag{3}$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{4}$$

$$(x^m)^n = x^{mn} \tag{5}$$

Let's show why (1) is true. Let m and n be 4 and 6 respectively. Then our problem would be,

$$x^4 x^6$$

We can now write this in expanded form as,

$$(x \cdot x \cdot x \cdot x \cdot) \cdot (x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot)$$

All we did is write out our original problem in expanded form. Now we can just count how many x's we have that we are multiplying. Because there are 10 x's we can write this as,

$$x^{10}$$

This is in accordance with (1) because,

$$x^4 x^6 = x^{4+6} = x^{10}$$

This is why we add exponents when we are multiplying their bases.

Let's now show why (2) is true. Let m and n be 5 and 3 respectively. Then our problem would be,

$$\frac{x^5}{x^3}$$

We can write this in expanded form as,

$$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x}$$

We know that when we have a fraction and the same number or variable is on the top and bottom we can cancel them out. Let's do that here.

$$\frac{\cancel{x \cdot x \cdot x \cdot} x \cdot x}{\cancel{x \cdot x \cdot x}}$$

We are left with 2 x's and so our answer is,

$$x^2$$

This is accordance with (2) because,

$$\frac{x^5}{x^3} = x^{5-3} = x^2$$

This is why we subtract exponents when we divide their bases.

To show why (3) is true let n be 3. Our problem is then,

$$(xy)^3$$

In expanded form this is,

$$xy \cdot xy \cdot xy = x \cdot x \cdot x \cdot y \cdot y \cdot y$$

Counting x's and y's yields,

$$x^3y^3$$

This is in accordance with (3) because,

$$(xy)^3 = x^3y^3$$

To show why (4) is true let n be 4. Our problem is then,

$$\left(\frac{x}{y}\right)^4$$

In expanded form this becomes,

$$\left(\frac{x}{y}\right) \cdot \left(\frac{x}{y}\right) \cdot \left(\frac{x}{y}\right) \cdot \left(\frac{x}{y}\right)$$

By multiplying these 4 fractions we obtain,

$$\frac{x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y}$$

Counting x's and y's yields,

$$\frac{x^4}{y^4}$$

This is in accordance with (4) because,

$$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$$

To show why (5) is true let m and n be 3 and 2 respectively. Our problem then becomes,

$$(x^3)^2$$

In expanded form this becomes,

$$x^3 \cdot x^3$$

We know from the first rule, however, that when multiplying two terms with the same bases we add their exponents. Thus, this becomes,

$$x^3 \cdot x^3 = x^6$$

This is in accordance with (5) because,

$$(x^3)^2 = x^{3 \cdot 2} = x^6$$

Two more general rules to remember are,

$$x^0 = 1 \tag{6}$$

$$x^{-n} = \frac{1}{x^n} \tag{7}$$

Now, lets do some examples.

Examples

- **Example 1**

$$(4x^2)(3x^6) \tag{8}$$

Because we are multiplying two factors with the same base (x) we will leave the bases and add the exponents. The 4 and 3 (the coefficients) will the be multiplied as usual.

$$= (4x^2)(3x^6) \tag{9}$$

$$= (4 \cdot 3)x^{2+6} \tag{10}$$

$$= 12x^8 \tag{11}$$

- **Example 2**

$$\frac{c^{-3}d^4}{c^{-6}d^{-3}} \tag{12}$$

Here, we are dividing exponents. We will subtract exponents who share the same bases. That means, c's from c's and d's from d's.

$$= \frac{c^{-3}d^4}{c^{-6}d^{-3}} \tag{13}$$

$$= c^{(-3-(-6))} \cdot d^{(4-(-3))} \tag{14}$$

$$= c^3 \cdot d^7 \tag{15}$$

- **Example 3**

$$(3u^3v^2)^2 \cdot (2uv^4)^{-4} \tag{16}$$

Here we we have factors being raised to a power. We will do that first. Remember from our exponent rules that when we raise an exponent to another exponent we multiply the two exponents. The numbers will be raised to the power as usual.

$$= (3u^3v^2)^2 \cdot (2uv^4)^{-4} \tag{17}$$

$$= (3^2 \cdot u^{(3\cdot 2)} \cdot v^{(2\cdot 2)}) \cdot (2^{-4} \cdot u^{(1\cdot -4)} \cdot v^{(4\cdot -4)}) \tag{18}$$

$$= (9u^6v^4) \cdot \left(\frac{1}{16}u^{-4}v^{-16}\right) \tag{19}$$

Now we just have to multiply factors with the same bases and add their exponents. u's with u's and v's with v's. We will also just multiply the numbers out in front as usual.

$$= (9u^6v^4) \cdot \left(\frac{1}{16}u^{-4}v^{-16}\right) \tag{20}$$

$$= (9 \cdot \frac{1}{16}) \cdot u^{(6+(-4))} \cdot v^{(4+(-16))} \tag{21}$$

$$= \frac{9}{16}u^2v^{-12} \tag{22}$$

We're almost done but we typically simplify so that our problem no longer has negative exponents. So we will do that here. Remember that we can flip a term from the top to the bottom (or visa versa) and the exponent will change sings. We will do that here but only with the v term because that's the term with a negative exponent.

$$= \frac{9}{16}u^2v^{-14} \tag{23}$$

$$= \frac{\frac{9}{16}u^2}{v^{14}} \tag{24}$$

This can probably be written even better as,

$$\frac{9u^2}{16v^{14}} \tag{25}$$

And we're done!

- **Example 4**

$$\left(\frac{x^3y^5}{xy}\right) \cdot \left(\frac{x^3}{y^2}\right)^2 \tag{26}$$

In this example, we have exponents raised to exponents. We will deal with that first.

$$= \left(\frac{x^3y^5}{xy}\right) \cdot \left(\frac{x^3}{y^2}\right)^2 \tag{27}$$

$$= \left(\frac{x^3y^5}{xy}\right) \cdot \left(\frac{x^6}{y^4}\right) \tag{28}$$

Now we can choose to deal with the fractions first (and subtract exponents) or multiple across first (and add exponents). Either is fine. I will choose to do the latter first out of personal preference.

$$= \left(\frac{x^3y^5}{xy}\right) \cdot \left(\frac{x^6}{y^4}\right) \tag{29}$$

$$= \frac{x^9y^5}{xy^5} \tag{30}$$

We're almost done. We just have a fraction remaining so we shall subtract exponents.

$$= \frac{x^9y^5}{xy^5} \tag{31}$$

$$= x^8y^0 \tag{32}$$

$$= x^8 \cdot 1 \tag{33}$$

$$= x^8 \tag{34}$$

Here are some problems for you to do on your own with answers at the bottom of the page. Good luck!

$$\frac{x^3y^5}{xy^2} \tag{35}$$

$$\frac{(3abc)^2 \cdot (4a^2b^3c^{-1})^3}{4abc^4} \tag{36}$$

$$\frac{3w^{-2}v^{-1}}{4w^{-4}v^{-6}} \tag{37}$$

$$\frac{(x^{-2}y^{-4})^2}{xy} \tag{38}$$

answers in order: x^2y^3 $\frac{144a^7b^{10}}{c^5}$ $\frac{3}{4}w^2v^5$ $\frac{1}{x^5y^9}$