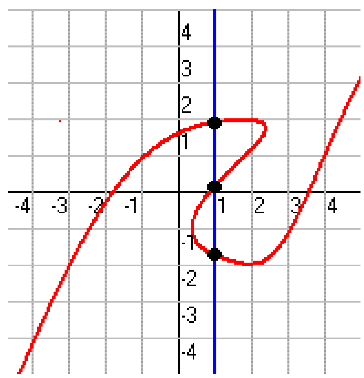
Graphs of Functions

There are lots of ways to visualize or picture a function in your head. You can think of it as a machine accepting inputs and shooting out outputs, or a set of ordered pairs, or whatever way you come up with.

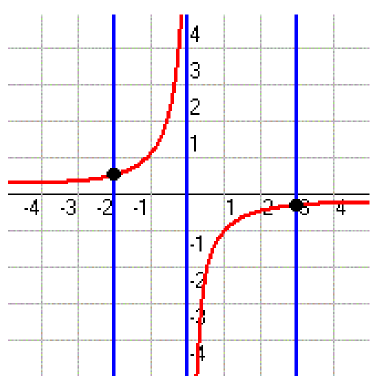
However, by far the most important way to visualize a function is through its graph. By looking at a graph in the xy-plane we can usually find the domain and range of the graph, discover asymptotes, and know whether or not the graph is actually a function.

The Vertical Line Test: A curve in the xy-plane is a function if and only if no vertical line intersects the curve more than once.

The Vertical Line Test allows us to know whether or not a graph is actually a function. Remember that a function can only take on **one** output for each input. We cannot plug in a value and get out two values. The Vertical Line Test will show this. For instance



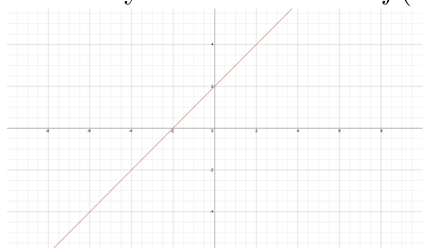
This red graph is NOT a function as it fails the Vertical Line Test in blue. We can draw a vertical line and it hits more than one point on our function. For this function in red when we plug in $x = 1$ we get 3 values out. And we know that a function can only have one output for each input. Thus, it isn't a function. However, this graph,



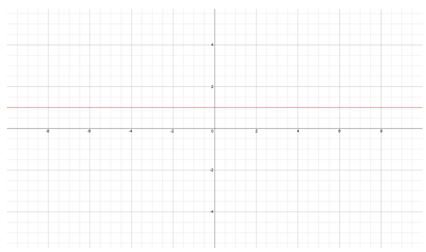
is a function as we can draw any vertical line (blue) and it does not intersect the function (red) at more than one point.

Now, let's at just some basic functions and their graphs. The following functions are very common in most math classes and it's probably a good idea if you can just memorize what their general shape is. It'll make a world of difference if you can picture what a basic function's graph looks like.

Linear Functions: Linear function only have at most a degree of 1. This means it has at most one x raised to a power of 1. They follow the form: $f(x) = mx + b$.

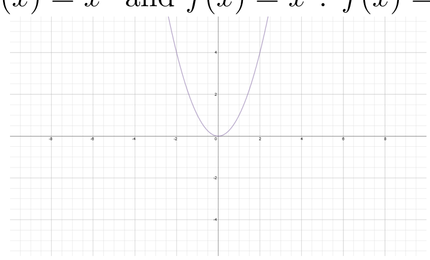


Linear functions (almost) *always* have infinite domains and ranges. The exception is when the graph is a horizontal line. This happens for functions that equal a constant such as $f(x) = b$. These functions have infinite domains but a range that has only one value, b. For instance,

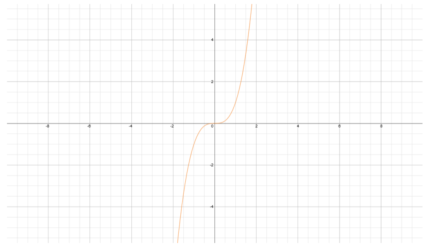


This function still has an infinite domain because it's defined everywhere but its range is only $y = 1$.

Power Functions: Power functions are functions that have a leading term greater than one. The most common are $f(x) = x^2$ and $f(x) = x^3$. $f(x) = x^2$ looks like,

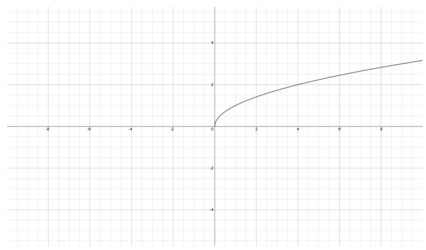


This function has an infinite domain (we can plug in any value of x) but its range is $R : [0, \infty)$. (we never dip below the x-axis because when we square a number it always becomes positive). $f(x) = x^3$ looks like,



In this case, we have an infinite domain and range.

Root Functions: Root functions are function that involve roots, square or otherwise. The most common is , $f(x) = \sqrt{x}$.



Square root functions have limited domains and ranges. Because we can never take the square root of a negative number, our domain is $D : [0, \infty)$ and likewise our range will also be, $R : [0, \infty)$. The great thing about graphs is we can also this visually!

Finally, we can also shift these basic graphs around using transformations. Transformation graphs look similar to their base graph but are different in some way. If f(x) is a base function then F(x) is our transformed function.

$$F(x) = -af(-bx + c) + d \tag{1}$$

Function transformations follow order of operations. We look inside the parenthesis first.

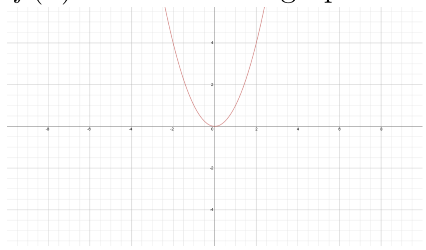
- **Inside f(x): Horizontal Changes**
  Step 1: The c term will give us a horizontal shift, left or right.
  Step 2: The - sign will reflect our graph across y-axis.
  Step 3: The b term gives rise to a horizontal stretch or compression.

- **Outside f(x): Vertical Changes**
  Step 4: This - sign will result in a vertical reflection across the x-axis.
  Step 5: The a term will give the function a vertical stretch or compression.
  Step 6: The d gives the graph a vertical shift up or down.

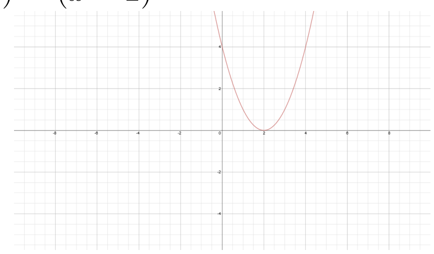
Let's look at an example.

- **Example 1**:
  Graph the following function: $F(x) = -(x - 2)^2 - 3$.

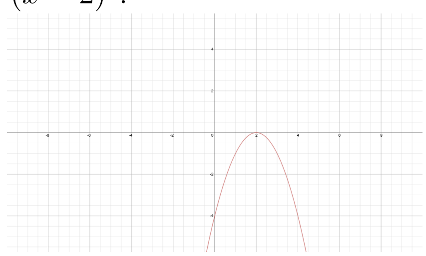
The base function here is $f(x) = x^2$. We will graph that first.



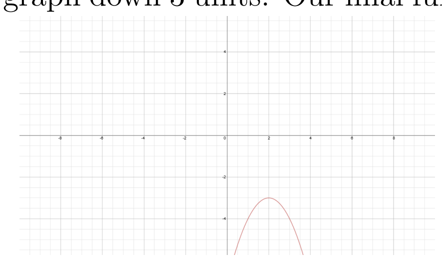
The first transformation we have to do is the horizontal shift to the right 2 units. This function will now be $f(x) = (x-2)^2$



We next account for the negative out in front which will flip our graph across the x-axis. This function is $f(x) = -(x-2)^2$.



Finally, we will shift our graph down 3 units. Our final function is $F(x) = -(x-2)^2 - 3$.



Our domain and range are $D : (-\infty, \infty)$ and $R : (-\infty, 3]$.

Here are some functions for you to graph on your own. First find the base function and then use our transformation rules to obtain the final graph then state the domain and range.

$$f(x) = -x^3 - 4 \tag{2}$$

$$g(x) = \sqrt{x-4} + 9 \tag{3}$$

$$h(x) = \frac{1}{x-4} \tag{4}$$

$$m(x) = \frac{1}{(x-4)^2} + 1 \tag{5}$$