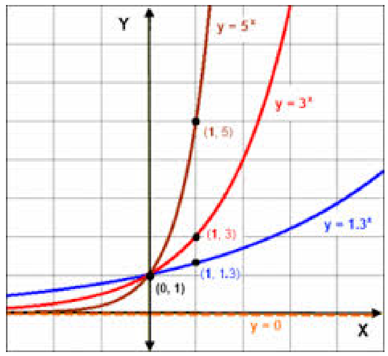
Exponentials and Logarithms

An exponential function is any function of the form,

$$f(x) = a^x \quad a \in \mathbb{R} \tag{1}$$

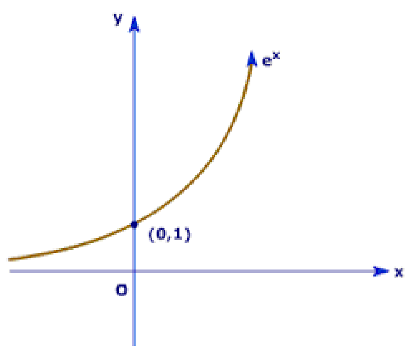
here, a is just any number being raised to a variable exponent. Exponential graphs look like,



Depending on how large a is the function will 'explode' up to infinity at different rates.

By far, the most common exponential is the number e. e is an irrational number and therefore goes on forever but it is approximately equal to 2.71. While e seems sort of arbitrary and pulled out of thin air, it does seem to have an eerie tendency to pop up in tons of real world applications.

The graph of $f(x) = e^x$ looks like,



The key things to notice is that $f(x)$ has a domain, $D : (-\infty, \infty)$ and a range, $R : (0, \infty)$. This implies that $f(x)$ has a horizontal asymptote, $y = 0$. e^x will get infinitesimally close to the line $y = 0$ but it'll never actually reach that line. Thus it has NO x intercepts. It does have a y intercept and it is at the point $(0, 1)$. This means that $e^0 = 1$.

Like all functions, exponential functions have inverses. The inverse of the exponential is the logarithm, or log, for short.

The logarithmic function with base a denoted by log_a is defined as,

$$log_a x = y \Leftrightarrow a^y = x \tag{2}$$

What this means is that $log_a x$ is the *exponent* that the base a must be raised to, to give x. When $a = e$ we say that the logarithm $log_e x$ is the *natural log* and we write it instead as $ln(x)$. We don't write the e. By writing ln we are implying the base is e.

Because the exponential function $f(x) = e^x$ and the natural log function $g(x) = ln(x)$ are inverses, we know that they are symmetrical about the line $y = x$.

We can summarize the following characteristics here:

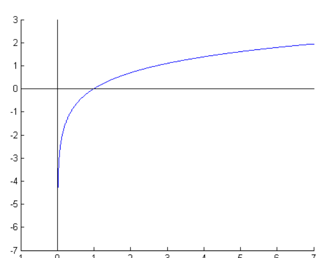
	Exponential Function	Logarithmic Function
Domain	$(-\infty, \infty)$	$(0, \infty)$
Range	$(0, \infty)$	$(-\infty, \infty)$
x - intercept	none	$(1,0)$
y - intercept	$(0,1)$	none
asymptote	$y = 0$	$x = 0$

With just this information, we can now apply transformations to the graphs of exponentials and logarithms like we do to usual functions. Here are some examples.

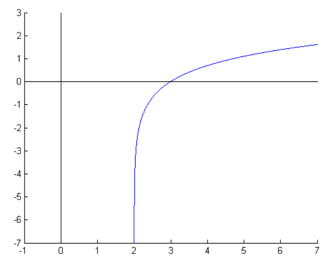
- **Example 1**: Graph the function and find its domain and range.

$$f(x) = -ln(x - 2) + 1 \tag{3}$$

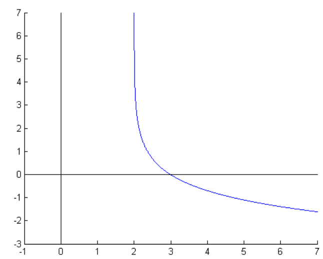
As usual we first recognize that our base function is $f(x) = ln(x)$. We will graph that first.



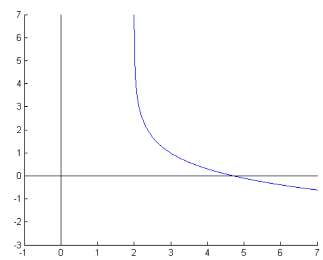
Next, we will apply the changes inside the parenthesis and shift our graph two units to the right, $f(x) = ln(x - 2)$.



We now deal with the negative out in front which will give us a vertical reflection about the x axis, $f(x) = -ln(x-2)$.



And finally we shift our graph up 1 unit, $f(x) = -ln(x-2)+1$.



For our final graph we see that our domain and range are going to be,

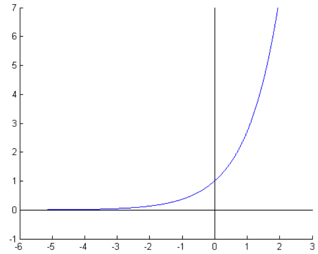
$$D : (2, \infty) \tag{4}$$
$$R : (-\infty, \infty) \tag{5}$$

Because our domain is shifted two units to the right our vertical asymptote is also shifted from $x = 0$ to $x = 2$.

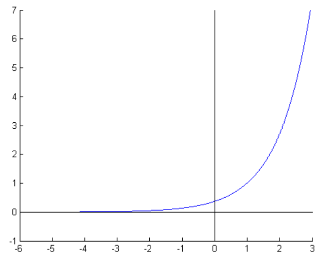
- **Example 2**: Graph the function and find its domain and range.

$$f(x) = -e^{x-1} - 2 \tag{6}$$

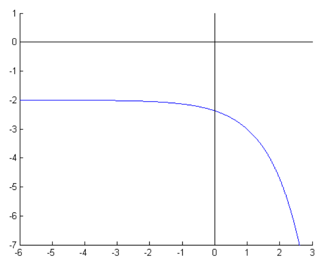
Our base function is $f(x) = e^x$ so we will graph that first.



We next deal with what first acts on our function which will be the $x-1$ in the exponent, $f(x) = e^{x-1}$. It shifts our function one unit to the right.



Next, the negative out in front flips our graph across the x axis and finally we bump our function two units down, $f(x) = -e^{x-1} - 2$.



We see that our domain and range, respectively are,

$$D : (-\infty, \infty) \tag{7}$$
$$R : (-\infty, -2) \tag{8}$$

This domain implies that our new horizontal asymptote is $y = -2$ and not $y = 0$.

Here are some example functions for you to work out on your own.

- **Problems**: Graph the following functions and find their domain and range.

$$f(x) = ln(x-4) - 1 \tag{9}$$
$$g(x) = e^{-x} + 2 \tag{10}$$
$$h(x) = ln(-x+4) \tag{11}$$
$$p(x) = -ln(x+2) - 6 \tag{12}$$