Solving Quadratic Equations

A quadratic equation is any equation of the form,

$$ax^2 + bx + c = 0 \tag{1}$$

Quadratic equations pop up so often in so many different applications and disciplines in the real world that we give them special attention. Projectiles, profits, peak reaction times in chemical reactions, the list truly could go on forever. Quadratics literally make up our world.

When we say "solve a quadratic equation" what we mean is that we want to find what value(s) of x, that when plugged into (1) makes the equation true. There are two ways to do this.

Factoring: When we factor a quadratic equation we are breaking it down into simpler parts. For instance if our equation is,

$$x^2 - x - 12 \tag{2}$$

we can basically guess and check our way to the factored form:

$$(x - 4)(x + 3) \tag{3}$$

We know this is true because if we foil out (3) we arrive back at (2).

However, there is a more systematic way and it is as followed: We try and pick two numbers that when multiplied equal our c term and add to be our b term. For instance, in our previous example, we chose two numbers, -4 and 3. Notice that these two numbers multiply to be $c = -12$ and add to be $b = -1$. Let's look at another example.

$$x^2 - 11x + 18 \tag{4}$$

To factor this we need to find two numbers that add to -11 and multiply to be 18. It's typically wise to think of two numbers that multiply to be the c term as there will be less combinations than those that add to be the b term. So, let's list all numbers that multiple to be 18.

$$18 = 18 \cdot 1 \tag{5}$$
$$18 = 6 \cdot 3 \tag{6}$$
$$18 = 9 \cdot 2 \tag{7}$$

These are all pairs of numbers that multiple to 18. However, none of them seem to be adding to -11. But notice, that all these pairs can be negative as well. This means,

$$18 = -18 \cdot -1 \tag{8}$$
$$18 = -6 \cdot -3 \tag{9}$$
$$18 = -9 \cdot -2 \tag{10}$$

We notice by inspection that $-9 + -2 = -11$ which our b term. Thus we can factor our problem so that,

$$x^2 - 11x + 18 = (x - 9)(x - 2) \tag{11}$$

Here are some example problems for you to work out on your own: (they are all factorable)

$$x^2 - 4x - 32 \tag{12}$$
$$x^2 + 11x + 28 \tag{13}$$
$$x^2 + 3x + 2 \tag{14}$$
$$x^2 - 2x + 1 \tag{15}$$
$$x^2 - 10x + 21 \tag{16}$$

Quadratic Formula: You should always try and factor first when given a quadratic equation. Sometimes though, the two numbers aren't always integers and are fractions. While it is still possible to factor in these cases it becomes increasingly difficult. This is where the quadratic equation is useful. The quadratic equation is as followed:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{17}$$

If you're only going to memorize one thing in math, let this be it. It is beyond useful. Let's use it.

$$x^2 + 8x + 14 \tag{18}$$

In this case, our coefficients are $a = 1$, $b = 8$, and $c = 14$. I cannot think of two integers that multiple to 14 and add to 8. So we use the quadratic formula.

$$x = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 1 \cdot 14}}{2 \cdot 1} \tag{19}$$
$$x = \frac{-8 \pm \sqrt{64 - 56}}{2} \tag{20}$$
$$x = \frac{-8 \pm \sqrt{8}}{2} \tag{21}$$

Thus we have two solutions. Namely,

$$x = \frac{-8 + \sqrt{8}}{2} \quad x = \frac{-8 - \sqrt{8}}{2} \tag{22}$$

Let's now look at a real world application where the quadratic equation becomes extremely useful.

Profit: You are in the business of selling trips for people to travel into space and orbit the Earth. Your costs for manufacturing/maintaining the space shuttles, paying your pilots, advertising, etc. are \$10,000,000. There is an additional cost of \$2,550 for every person that buys a trip from you and that you send into orbit. Your financial team tells you that that the function that models your unit sales based on price is $unit\,sales = 1,000,000 - 3,500p$. p is price. This means that based on what you charge people for a ticket the amount of tickets sold will also change. This makes sense. If you charge a lot, you'll typically sell less. So the question now becomes: What price should you sell your tickets to maximize your profits?

Solution:
Remember, the unit sales is just how *many* trips you sell. To find the total amount of sales, we will multiply the unit sales with the price in which you sell your trips.

$$\begin{align} Total\,Sales = Unit\,Sales \cdot Price &= (1,000,000 - 3,500p) \cdot p \tag{23} \\ &= 1,000,000p - 3,500p^2 \tag{24} \end{align}$$

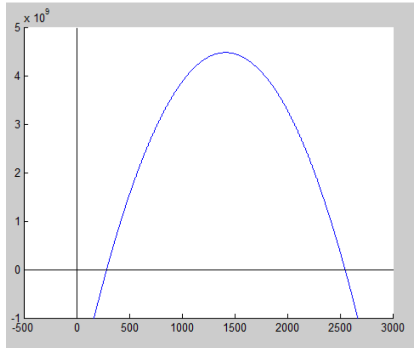
This value is what we call revenue. It's how much money we will bring in, in total. But we also have costs. Our cost formula will be:

$$\begin{align} Total\,Costs &= 10,000,000 + 2,550(1,000,000 - 3,500p) \tag{25} \\ &= 10,000,000 + 2,550,000,000 - 8,925,000p \tag{26} \\ &= 2,560,000,000 - 8,925,000p \tag{27} \end{align}$$

We now know that that $Profit = Revenue - Costs$. Thus, our profit formula is:

$$\begin{align} Profit &= Total\,Sales - Total\,Costs \tag{28} \\ &= 1,000,000p - 3,500p^2 - (2,560,000,000 - 8,925,000p) \tag{29} \\ &= 1,000,000p - 3,500p^2 - 2,560,000,000 + 8,925,000p \tag{30} \\ &= -3,500p^2 + 9,925,000p - 2,560,000,000 \tag{31} \end{align}$$

You finally have your profit formula (the little p is price!). This is what you wish to maximize. Let's graph this function and see what you can learn from it.



On this graph the x-axis represents the amount of tickets sold. The y-axis is then the profit. So what this graph is telling you, is that at the peak, or maximum, you will have a maximum profit. The x-value, or the amount of tickets sold, will be the optimal amount you should sell to obtain the maximum profit.

Ok, so now the quadratic formula comes in. The two values that the quadratic formula give you are the zeros, or x intercepts, of a graph. These two values lie on either side of the maximum (you can see this from the graph as well). Thus, we will use the quadratic formula to find the zeros of the function, find the average of the two, and that value will be where our maximum profit occurs.

$$p = \frac{-9,925,000 \pm \sqrt{(9,925,000^2) - 4 \cdot (-3,500) \cdot (-2,560,000,000)}}{2 \cdot -3,500} \tag{32}$$

$$= \frac{-9,925,000 \pm 7,916,162}{-7000} \tag{33}$$

We have two (rounded) solutions (which is normal):

$$p_1 = 286 \quad p_2 = 2548 \tag{34}$$

These are our two x-intercepts. We will take the average of the two values to find where our maximum profit occurs.

$$p = \frac{286 + 2548}{2} \tag{35}$$

$$= 1417 \tag{36}$$

Thus, you should sell 1417 tickets a year to maximize your profit. By looking at the graph it would appear that this is where the maximum does in fact occur.

Here is a problem for you to work on your own:

You and some friends are cliff diving in the Jemez Wilderness. Your height above the ground (in feet) as a function of time can be modeled by the equation $h(t) = -16t^2 + 16t + 75$. After how many seconds did you hit the water?
ANSWER: Approximately 2.71 seconds.